

ON THE RENORMALIZATION GROUP FOR THE INTERACTING MASSIVE SCALAR FIELD THEORY IN CURVED SPACE

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Abstract. The effective action for the interacting massive scalar field in curved space-time is derived using the heat-kernel method. Starting from this effective action, we establish a smooth quadratic form of the low-energy decoupling for the four-scalar coupling constant λ and for the nonminimal interaction parameter ξ . The evolution of this parameter from the conformal value $1/6$ at high energies down to the IR regime is investigated within the two toy models with negative and positive four-scalar coupling constants.

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1 Introduction

The relevance of scalar fields is significant in both particle physics and cosmology. For instance, the Higgs scalar is in the heart of the Standard Model of particle physics (SM) providing the possibility of the Spontaneous Symmetry Breaking and the Higgs mechanism. Furthermore, many extensions of the SM imply new scalars corresponding either to the new symmetries (e.g. supersymmetry) or to their breaking. In cosmology the scalar fields are invoked to mimic the variable vacuum energy. In particular, they represent an important element of the inflaton models and also, as different sorts of quintessence, may play the role of the Dark Energy in the late Universe. Obviously, the cosmological applications of scalar fields indicate the special importance of their interaction to gravity. It is well-known that there is an arbitrary element called a non-minimal term. Say, the Lagrangian of a single scalar φ includes, along with the covariant kinetic and massive terms $(\nabla\varphi)^2 + m^2\varphi^2$, the non-minimal term $\xi R\varphi^2$, where R is the scalar curvature and ξ is the parameter of the non-minimal interaction. This term plays a special role, because it represents the unique possible non-minimal structure with the dimensionless coefficient. All other terms include coefficients with the inverse-mass dimensions and therefore will be strongly suppressed at

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low energies. In this situation, any information concerning the value of ξ would be significant for the applications.

In classical theory the value of ξ is arbitrary, but at the quantum level one can impose some constraints. In the massless case $m = 0$, the value $\xi = 1/6$ corresponds to the theory with the local conformal symmetry. Only in this case the trace of the Energy-Momentum Tensor for the scalar field equals zero. On the other hand, we know that the Energy-Momentum Tensor of a massless particle has zero trace. Therefore, the value $\xi = 1/6$ is the only choice which provides a correspondence between field and particle descriptions in the massless limit (see, e.g. [1]). The massless limit means, in particular, that the equation of state for the ideal gas of the corresponding particles is $\rho = 3p$. Of course, any gas of identical massive particles will approach this equation of state when the kinetic energies of these particles become much greater than their masses. Therefore, we can suppose that the UV limit for the scalar field corresponds to $\xi = 1/6$. Then, in order to learn the value of ξ at the lower energies we have to start from the conformal point $\xi = 1/6$ in the UV. Furthermore, Quantum Field Theory provides a natural mechanism for calculating the values of ξ at the lower energies through the Renormalization Group equations formulated in curved space-time [2, 3, 4]. In the early works [2, 3] on the subject (see the review and many other references in [4]) the main attention has been paid to the renormalization group running of ξ in various gauge models. The remarkable achievements were the discovery of the models with the UV stable conformal fixed point [5] and the systematic investigation of the possibility of such fixed point [6]. The next step has been done in [7], where the problem of how the system evolves *starting from* the conformal fixed point has been carefully studied, taking into account the possible quantum effects of the conformal factor [8] of the metric and the higher-loop effects. Another framework for taking the quantum gravity effects into account is the higher derivative quantum gravity [9], where the renormalization group equation for ξ gains additional contributions from the gravitational field loops.

The common point of all mentioned approaches is that they are based on the most simple Minimal Subtraction scheme of renormalization. On the other hand, despite this scheme is very efficient in the UV corner of the theory, it is not really trustable when we intend tracing back the running from the massless UV fixed point to the intermediate or even IR regime. In this case the masses of the particles become important, because they modify the Renormalization Group equations (RG). An alternative approach, based on the more physical mass-dependent renormalization scheme (see, e.g. [10] for the introduction and further references), has been applied to the gravitational problems just recently [11, 12]. The focus of attention of these works was the renormalization group and decoupling in the vacuum sector of the quantum field theory in an external classical gravitational background. It turned out, that the physical renormalization group can be formulated in the framework of the linearized gravity. The low-energy decoupling of the massive scalar [11], fermion and vector fields (including the QCD constituents) [12] performs smoothly, similar to the Appelquist and Carazzone theorem in QED [13]. In the present letter we apply the same method to the analysis of the renormalization group for the non-minimal parameter

ξ . For this end we consider the theory with the action

$$S = \int d^4x \sqrt{|g|} \left\{ \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \xi R \phi^2 - \frac{\lambda}{24} \phi^4 \right\} \quad (1)$$

and derive both divergent and finite parts of the one-loop effective action in the matter field sector. In principle, such calculation can be performed using the Feynman diagrams [11], but since the heat-kernel technique [14, 15] provides the possibility to perform the calculation in the most economic way, we shall use this method in the way adapted in [11, 12].

The paper is organized as follows. In the next section we derive the one-loop effective action. In section 3 we obtain the β -functions for the coupling λ and the non-minimal parameter ξ , and establish the explicit form of their decoupling at low energies. Furthermore, we discuss the scale dependence of λ and ξ . In section 4 we draw our conclusions and outline the prospects for the future work.

2 Derivation of Effective Action

The one-loop Euclidean effective action corresponding to the theory (1) is given by

$$\bar{\Gamma}^{(1)} = -\frac{1}{2} \text{Tr} \ln \left[-\hat{1} \square - \hat{1} m^2 - \hat{P} + \frac{\hat{1}}{6} R \right], \quad (2)$$

where the hats indicate operators acting in the space of the quantized fields. As far as we are going to consider only the scalar field φ , for us $\hat{1} = 1$, where 1 is the image of the delta function. Furthermore, $\hat{P} = -(\xi - 1/6)R + \lambda\phi^2/2$. The effective action (2) can be expressed by means of the proper time integration of the heat kernel

$$\bar{\Gamma}^{(1)} = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} K(s). \quad (3)$$

As usual, the last expression can be expanded into the powers of the field strengths (curvatures), $R_{\mu\nu\alpha\beta}$ and \hat{P} . We would like to remind that the one-loop effective potential contains arbitrary powers of the field ϕ but (by definition) not its derivatives. In the present case we expand the effective action up to the fourth order in ϕ but take into account an arbitrary number of derivatives. The derivative expansion of the one-loop effective action with an arbitrary powers of ϕ has been considered earlier in [16].

Up to the second order in curvatures, the expansion has the form [14, 15]

$$\begin{aligned} \text{Tr} K(s) = & \frac{\mu^{4-2\omega}}{(4\pi s)^\omega} \int d^{2\omega-4}x \sqrt{g} e^{-sm^2} \text{tr} \left\{ \hat{1} + s\hat{P} + s^2 [R_{\mu\nu} f_1(-s\square) R^{\mu\nu} + \right. \\ & \left. + R f_2(-s\square) R + \hat{P} f_3(-s\square) R + \hat{P} f_4(-s\square) \hat{P}] \right\}. \end{aligned} \quad (4)$$

Here ω is the dimensional parameter, μ is an arbitrary renormalization parameter with dimension of mass and the functions f_i are given by

$$f_1(\tau) = \frac{f(\tau) - 1 + \tau/6}{\tau^2}, \quad f_2(\tau) = \frac{f(\tau)}{288} + \frac{f(\tau) - 1}{24\tau} - \frac{f(\tau) - 1 + \tau/6}{8\tau^2},$$

$$f_3 = \frac{f(\tau)}{12} + \frac{f(\tau) - 1}{2\tau}, \quad f_4 = \frac{f(\tau)}{2}, \quad \text{where} \quad f(\tau) = \int_0^1 d\alpha e^{\alpha(1-\alpha)\tau}, \quad \tau = -s\Box.$$

After straightforward calculations [12], using the notations $t = sm^2$, $u = \tau/t = -\Box/m^2$, we obtain an explicit expression for the Effective Action up to the second order in \hat{P} and $R_{\mu\nu}$

$$\begin{aligned} \bar{\Gamma}^{(1)} = & -\frac{1}{2(4\pi)^2} \int d^{2\omega-4}x \sqrt{g} \left(\frac{m^2}{4\pi\mu^2} \right)^{\omega-2} \int_0^\infty dt e^{-t} \left\{ \frac{m^4}{t^{\omega+1}} - \frac{m^2}{t^\omega} \left(\xi - \frac{1}{6} \right) R \right. \\ & \left. + \frac{m^2}{4t^\omega} \lambda \phi^2 + \sum_{i=1}^5 l_i^* R_{\mu\nu} M_i R^{\mu\nu} + \sum_{j=1}^5 l_j R M_j R + \sum_{k=1}^5 l_k^\xi R M_k \phi^2 + \sum_{n=1}^5 l_n^\lambda \phi^2 M_n \phi^2 \right\}, \end{aligned} \quad (5)$$

where (we correct a misprint of [12] in the expression for M_1)

$$M_1 = \frac{f(tu)}{t^{\omega-1}}, \quad M_2 = \frac{f(tu)}{t^\omega u}, \quad M_3 = \frac{f(tu)}{t^{\omega+1} u^2}, \quad M_4 = \frac{1}{t^\omega u}, \quad M_5 = \frac{1}{t^{\omega+1} u^2},$$

and the coefficients have the form

$$\begin{aligned} l_{1,2}^* &= 0, \quad l_3^* = 1, \quad l_4^* = \frac{1}{6}, \quad l_5^* = -1, \\ l_1 &= \frac{1}{288} - \frac{1}{12} \left(\xi - \frac{1}{6} \right) + \frac{1}{2} \left(\xi - \frac{1}{6} \right)^2, \quad l_2 = \frac{1}{24} - \frac{1}{2} \left(\xi - \frac{1}{6} \right), \\ l_3 &= -\frac{1}{8}, \quad l_4 = -\frac{1}{16} + \frac{1}{2} \left(\xi - \frac{1}{6} \right), \quad l_5 = \frac{1}{8}, \\ l_1^\xi &= -\frac{\lambda}{2} \left(\xi - \frac{1}{6} \right) + \frac{\lambda}{24}, \quad l_2^\xi = \frac{\lambda}{4}, \quad l_{3,5}^\xi = 0, \quad l_4^\xi = -\frac{\lambda}{4}, \\ l_1^\lambda &= \frac{\lambda^2}{8}, \quad l_{2,3,4,5}^\lambda = 0. \end{aligned}$$

Indeed, the coefficients l_i and l_j^* are the same as for the free scalar field [11], while the last two sets are completely due to the interaction and do not have analogs in the free field case. At this point one has to perform the integration in the variables t and α . The result can be expressed in terms of notations [11]

$$A = 1 - \frac{1}{a} \ln \frac{1+a/2}{1-a/2}, \quad a^2 = \frac{4\Box}{\Box - 4m^2}. \quad (6)$$

The calculation of the integrals needs the expansions

$$\left(\frac{m^2}{4\pi\mu^2} \right)^{\omega-2} = 1 + (2-\omega) \ln \left(\frac{4\pi\mu^2}{m^2} \right) + \dots \quad (7)$$

and

$$\begin{aligned} \Gamma(2-\omega) &= \frac{1}{2-\omega} - \gamma + \mathcal{O}(2-\omega), \\ \Gamma(1-\omega) &= -\frac{1}{2-\omega} + \gamma - 1 + \mathcal{O}(2-\omega), \\ \Gamma(-\omega) &= \frac{1}{2(2-\omega)} - \frac{\gamma}{2} + \frac{3}{4} + \mathcal{O}(2-\omega). \end{aligned}$$

The integrals of expressions $e^{-t}M_i$ are listed below:

$$\begin{aligned}
\int_0^\infty dt e^{-t} M_1 &= \frac{1}{\epsilon} + 2A + \mathcal{O}(2 - \omega), \\
\int_0^\infty dt e^{-t} M_2 &= \left(\frac{1}{12} - \frac{1}{a^2} \right) \left(\frac{1}{\epsilon} + 1 \right) - \frac{4A}{3a^2} + \frac{1}{18} + \mathcal{O}(2 - \omega), \\
\int_0^\infty dt e^{-t} M_3 &= \left(\frac{1}{2a^4} - \frac{1}{12a^2} + \frac{1}{60} \right) \left(\frac{1}{\epsilon} + \frac{3}{2} \right) + \frac{8A}{15a^4} - \frac{7}{180a^2} + \frac{1}{400} + \mathcal{O}(2 - \omega), \\
\int_0^\infty dt e^{-t} M_4 &= \frac{a^2 - 4}{4a^2} \left(\frac{1}{\epsilon} + 1 \right) + \mathcal{O}(2 - \omega), \\
\int_0^\infty dt e^{-t} M_5 &= \frac{(a^2 - 4)^2}{32a^4} \left(\frac{1}{\epsilon} + \frac{3}{2} \right) + \mathcal{O}(2 - \omega),
\end{aligned} \tag{8}$$

where we denoted

$$\frac{1}{\epsilon} = \frac{1}{2 - w} + \ln \left(\frac{4\pi\mu^2}{m^2} \right) - \gamma.$$

Let us notice that, different from the previous publications [11, 12], we included the Euler constant into (8) explicitly. Indeed, this does not change the final result for the effective action or for the β -functions, because this constant can be always removed by the change of the renormalization parameter μ . After some algebra, we arrive at the expression for the one-loop effective action

$$\begin{aligned}
\bar{\Gamma}_{scalar}^{(1)} &= \frac{1}{2(4\pi)^2} \int d^4x g^{1/2} \left\{ \frac{m^4}{2} \cdot \left(\frac{1}{\epsilon} + \frac{3}{2} \right) + \left(\xi - \frac{1}{6} \right) m^2 R \left(\frac{1}{\epsilon} + 1 \right) \right. \\
&\quad + \frac{1}{2} C_{\mu\nu\alpha\beta} \left[\frac{1}{60\epsilon} + k_W(a) \right] C^{\mu\nu\alpha\beta} + R \left[\frac{1}{2\epsilon} \left(\xi - \frac{1}{6} \right)^2 + k_R(a) \right] R \\
&\quad \left. - \frac{\lambda}{2\epsilon} m^2 \phi^2 + \phi^2 \left[\frac{\lambda^2}{8\epsilon} + k_\lambda(a) \right] \phi^2 + \phi^2 \left[-\frac{\lambda}{2\epsilon} \left(\xi - \frac{1}{6} \right) + k_\xi(a) \right] R \right\}.
\end{aligned} \tag{9}$$

The expressions for $k_W(a)$ and $k_R(a)$ can be found in [12] and the new formfactors corresponding to the scalar self-interaction and non-minimal interaction between scalar and metric are

$$k_\lambda(a) = \frac{A\lambda^2}{4}, \tag{10}$$

$$k_\xi(a) = \lambda \left[\frac{A(a^2 - 4)}{12a^2} - \frac{1}{36} - A \left(\xi - \frac{1}{6} \right) \right], \tag{11}$$

where we used the notations (6). The two formfactors (10), (11) contain all information about the scale dependence of the parameters λ and ξ . In the next section we shall discuss this dependence in details.

3 Renormalization group equations and running parameters

In the modified Minimal Subtraction scheme ($\overline{\text{MS}}$) the β -function of the effective charge, C , is defined as

$$\beta_C(\overline{\text{MS}}) = \lim_{n \rightarrow 4} \mu \frac{dC}{d\mu}. \quad (12)$$

When we apply this procedure to the expression (9) the β -functions of λ and ξ , in the $\overline{\text{MS}}$ scheme, coincide with the usual ones, obtained in the local covariant approach [4]. The main advantage of the mass-dependent scheme is that the renormalization point where the subtraction is done ($p^2 = M^2$) has a particular physical meaning, giving us explicit information on the decoupling of massive fields in external gravitational background.

The β -function is defined, in the mass-dependent scheme, as $\lim_{n \rightarrow 4} M \frac{d}{dM}$ applied to the regularized form-factor of the corresponding term in the effective action. This procedure is equivalent to taking $-pd/dp$ of the formfactor of the ϕ^4 and $R\phi^2$ -terms. The β_λ -function which follows from this procedure has the form

$$\beta_\lambda = \frac{3\lambda^2}{4(4\pi)^2} (a^2 - a^2 A + 4A). \quad (13)$$

The UV limit corresponds to $a \rightarrow 2$ and gives

$$\beta_\lambda^{UV} = \frac{3\lambda^2}{(4\pi)^2}, \quad (14)$$

providing a perfect fit with the well-known result of the $\overline{\text{MS}}$ -scheme $\beta_\lambda^{UV} = \beta_\lambda^{\overline{\text{MS}}}$. In the IR limit $a \rightarrow 0$ we meet

$$\beta_\lambda^{IR} = \frac{\lambda^2}{2(4\pi)^2} \cdot \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right). \quad (15)$$

Clearly, this means the standard IR decoupling for the effective charge λ , similar to the Appelquist and Carazzone theorem in QED [13].

Let us use the β -function (13) to investigate the running of λ between the UV and IR limits⁴. The solution of the renormalization group equation

$$\mu \frac{d\lambda(\mu)}{d\mu} = \beta_\lambda^{\overline{\text{MS}}}, \quad \lambda(m) = \lambda \quad (16)$$

is well-known

$$\lambda(\mu) = \frac{\lambda}{1 - \frac{3\lambda}{(4\pi)^2} \ln(\mu/m)}. \quad (17)$$

The expression above is singular in the UV where it manifests the Landau pole for $\lambda > 0$ and also singular in the IR for $\lambda < 0$. The last property is because the $\overline{\text{MS}}$ -scheme is not sensitive to the presence of the mass of the field.

⁴We would like to point out that the β -function for m^2 is exactly zero at one loop.

It is remarkable that the renormalization group equation with the physical β_λ -function

$$p \frac{d\lambda(p)}{dp} = \beta_\lambda(\lambda, m, p), \quad \lambda(m) = \lambda \quad (18)$$

also admits analytic solution in terms of elementary functions

$$\lambda(p) = \lambda \left[1 - \frac{3\lambda}{(4\pi)^2} \left(\sqrt{5} \ln \frac{\sqrt{5}-1}{\sqrt{5}+1} + \frac{\sqrt{4m^2+p^2}}{p} \ln \frac{\sqrt{4m^2+p^2}+p}{\sqrt{4m^2+p^2}-p} \right) \right]^{-1}. \quad (19)$$

Qualitatively, the behaviours of the two functions (17) and (19) are not very different. In particular, both manifest the Landau pole, as can be seen at the Figure 1⁵. The main difference is the position of this pole which is moving farther into UV limit for the case of the physical renormalization group (19). Of course, from the physical point of view this difference is not very important, at least when we have in mind the Higgs field. The reason is that the Landau pole corresponds to extremely high energies. For example, if we assume that the scalar mass is 100 GeV and that $\lambda = 1$ (extreme case, for smaller values of λ the pole moves even to much higher energies), the position of the pole is at $p \approx 10^{23} \text{ GeV}$. This energy range is far beyond the Planck scale, and therefore it does not pose a problem for, e.g. the Standard Model. However, it makes a problem for us, because our main purpose is to evaluate a running of the parameter ξ from the conformal UV point $\xi = 1/6$ down to the IR regime. In this situation we decided to use the following two toy models: one with $\lambda \equiv 1$ (let us notice that the greater value of λ enforces the running of ξ) and another one with the negative sign of the coupling λ . The main disadvantage of the last model is that the classical potential of this theory is not bounded from below. However, there is a serious advantage - the asymptotically free (AF) behaviour in the UV. Let us emphasize that we consider the theory with $\lambda < 0$ just as a toy model which is called to mimic the renormalization group running of ξ in a general AF GUT-like theory. The derivation of the physical β -functions for the general GUT theory is much more involved and will be reported elsewhere. For the negative initial value of λ the behaviour of the corresponding effective charge is presented at the Figure 2.

For the β_ξ -function we arrive at the following result:

$$\beta_\xi = \frac{\lambda}{48(4\pi)^2 a^2} \left[(a^2 - 4)(a^2 A - 12 A - a^2) + 12a^2(4A + a^2 - a^2 A) \left(\xi - \frac{1}{6} \right) \right]. \quad (20)$$

It is easy to see that the last expression meets standard tests and expectations. In the UV limit $a \rightarrow 2$ there is correspondence with the known $\overline{\text{MS}}$ -scheme result

$$\beta_\xi^{UV} = \beta_\xi^{\overline{\text{MS}}} = \frac{\lambda}{(4\pi)^2} \left(\xi - \frac{1}{6} \right). \quad (21)$$

In the low-energy regime there is a standard quadratic decoupling of the massive field

$$\beta_\xi^{IR} = \frac{\lambda}{6(4\pi)^2} \left[\left(\xi - \frac{1}{6} \right) - \frac{1}{30} \right] \cdot \frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4}\right). \quad (22)$$

Thus, we have obtained the analog of the Appelquist and Carazzone theorem in the matter field sector for the theory of a massive scalar field in curved space-time. The decoupling takes place not only for the four-scalar coupling λ , but also for the nonminimal parameter ξ .

⁵All the plots have been obtained using the Mathematica program [17].

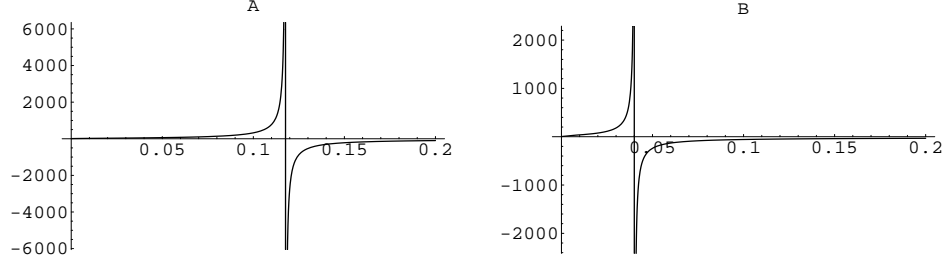


Figure 1: *Plots for the scale dependence for the effective charge (A) $\lambda(p/m)$ in the mass-dependent scheme of renormalization, and (B) $\lambda(\mu/m)$ in the $\overline{\text{MS}}$ -scheme. Both plots correspond to the initial data $\lambda(\mu = m) = 0.1$ and the units 10^{230} for both μ/m and p/m .*

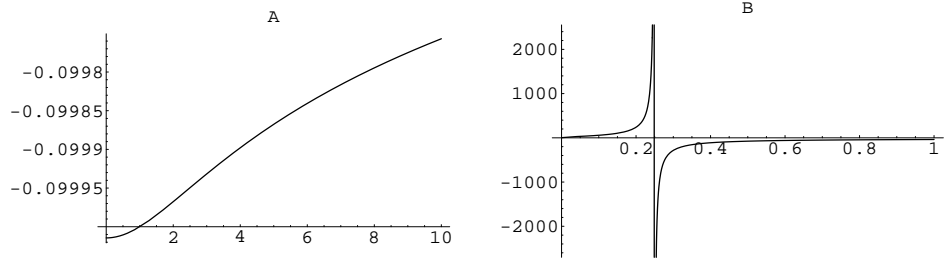


Figure 2: *The plots representing the scale dependence of λ for the AF model with $\lambda < 0$. Two cases are presented: (A) λ as a function of p/m for the mass-dependent scheme of renormalization); (B) λ as a function of μ/m for the $\overline{\text{MS}}$ -scheme. In the last case the units of μ/m are 10^{-228} . It is remarkable that the plot (A) shows regular behaviour in the IR, while the plot (B) has an IR pole, like for the massless theory. For $p > m$ the two plots are almost identical.*



Figure 3: *The relevant 1-loop diagrams in the $\lambda\varphi^4$ -theory. The straight lines correspond to the massive scalar and wavy lines to the external field $h_{\mu\nu}$. The formfactor (10) coming from the first diagram does not depend on the presence of external $h_{\mu\nu}$ lines, while for the formfactor (11) coming from the second diagram at least one of these external lines is necessary.*

Let us pay attention to the general form of the expression for the β_ξ -function (20). In the UV limit, this β -function contains the factor $(\xi - 1/6)$. This property of the β_ξ -function is not accidental, it is related to the conformal invariance of the one-loop divergences which can be proved in a general form [3, 4]. At the same time, out of the UV limit the β_ξ -function is not proportional to $(\xi - 1/6)$ due to the effect of the mass of the field. This is, indeed, the most important difference between β_ξ and $\beta_\xi^{\overline{\text{MS}}}$.

Using the momentum-subtraction scheme of renormalization, in the framework of linearized gravity [11] one can consider the renormalization group equation for the effective charge ξ

$$p \frac{d\xi(p)}{dp} = \beta_\xi(\lambda, m, p), \quad \xi(p_0) = \xi, \quad (23)$$

where the $\beta_\xi = \beta_\xi(\lambda, m, p)$ -function is given by the expression (20). The last equation describes the reaction of the effective charge ξ to the change of the momenta in the external loop of the corresponding Feynman diagram, shown at the Figure 3. For the sake of completeness, we have also presented a diagram which contributes to the β_λ -function.

Let us analyse the renormalization group equation (23). It is easy to see that (23) is nothing but a linear ordinary differential equation and therefore its analytic solution can be easily obtained in terms of integrals involving the expression (19). The formula which follows from this procedure is extremely bulky and hence it is useless for us. Therefore we turn to the qualitative and numerical considerations.

In the momentum-subtraction renormalization scheme the β -function is not directly linked to the divergences (except in the UV limit where the correspondence with the $\overline{\text{MS}}$ -scheme holds). That is why the conformal value $\xi = 1/6$ is not a fixed point for the theory of massive scalar field. At the same time, for the reasons explained in the Introduction, we have to assume the conformal value as the initial point of the renormalization group trajectory in the UV. Therefore the most natural option is to impose the renormalization condition $\xi(p_0) = 1/6$ in extreme UV $p_0^2 \gg m^2$ and integrate it numerically until the intermediate $p^2 \sim m^2$ or IR $p^2 \ll m^2$ regimes.

The plots of the numerical solutions of the (23) for different behaviours of $\lambda = \lambda(p/m)$ are presented at the Figure 4. As one can see at these plots, the deviation of ξ from the conformal value $1/6$ really takes place but the range of this deviation is not very big for the AF model with $\lambda(p = m) = -0.1$. In the extreme case of large and constant coupling $\lambda \equiv 1$, the value of ξ increases, approximately, 8 % between UV and IR. About the same range of running takes place

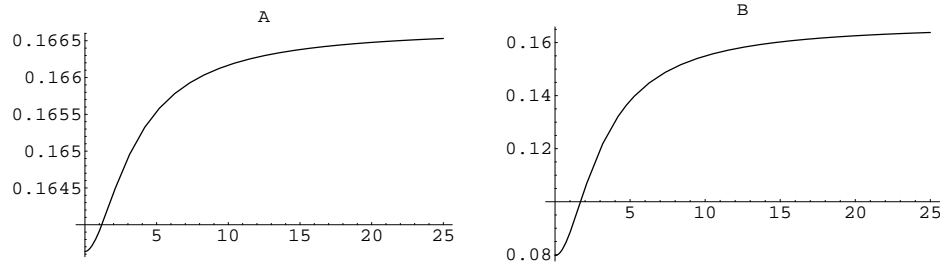


Figure 4: The plots of $\xi(p/m)$, corresponding to the initial value $\xi(p \gg m) = 1/6$ and two different behaviours (A) $\lambda(p = m) = -0.1$ (AF case with the unbounded from below potential) and (B) $\lambda \equiv 1$.

between UV and the point $p^2 = m^2$. After all, despite the conformal value $\xi = 1/6$ can not be exact in the massive scalar theory, it may serve as a reasonable approximation, at least in the UV region $p^2 > m^2$. The conformal approximation becomes very good for the AF theory with relatively weak coupling. Let us notice that the same conclusion has been obtained earlier in [7] within a very different approach, which involves IR quantum gravity as a quantum theory of the conformal factor of the metric [8].

4 Conclusions

We have applied the exact solution for the heat-kernel of the second-order minimal operator [14, 15] for the derivation of the effective action for the interacting massive scalar field. The methods which were developed earlier in [11, 12] made this calculation, technically, relatively simple and moreover open the possibility to derive the effective action of other interacting theories (e.g. Standard Model or GUT's) without using Feynman diagrams or, e.g. pinch technique. The last observation may be important, in particular, for the investigations related to the supersymmetry [18].

Using the effective action and the linearized gravity approach in curved space-time, we derived the renormalization group β -functions for the four-scalar coupling constant λ and for the nonminimal interaction parameter ξ . Both β -functions correspond to the physical momentum-subtraction scheme of renormalization and are essentially more complicated than the standard Minimal Subtraction β -functions for the same effective charges. In the UV limit, however, there is a perfect correspondence between the two sets of the β -functions. At low energies we observe the standard quadratic form of the decoupling for the scalar field, in accordance to the expectations based on the AC decoupling theorem [13].

The scale dependence of the four-scalar coupling constant λ does not depend on the presence of an external gravity field. Moreover, it is qualitatively the same as in the \overline{MS} renormalization scheme. In both cases there is a Landau pole, but for the momentum-subtraction scheme case the

position of this pole is shifted (about 3 times) to the UV compared to \overline{MS} case. If changing the sign of the coupling λ , the theory becomes asymptotically free. Indeed, the theory with $\lambda < 0$ is not a realistic model, because the scalar potential is not bounded from below. However, it may serve as a toy model for the investigation of the running of ξ . We traced the evolution of this parameter from the conformal value $1/6$ in the high energy limit down to the IR regime and found that the numerical change of ξ is not very big, in fact it does not exceed a few percents for the value $\lambda(p = m) = 0.1$. One expects a similar range of running within the realistic AF models like GUT's.

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